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Orthonormalized eigenstates of the operator $(a_q f(N_q))^k$ $(k \geqslant 1)$ and their generation

Xiao-Ming Liu

Department of Physics, Beijing Normal University, Beijing 100875, People's Republic of China and
CCAST (World Laboratory), PO Box 8730, Beijing 100080, People's Republic of China

E-mail: Liuxm@263.net

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Abstract. The k orthonormalized eigenstates of the kth power $(a_q f(N_q))^k$ $(k \geqslant 1)$ of the generalized q-boson annihilation operator $a_q f(N_q)$ are obtained, and their properties are discussed. An alternative method to construct them is proposed, and it is shown that all of them can be expressed as a linear superposition of k q–f-coherent states that have the same amplitude but different phases. Physically, they can be generated by a linear superposition of the time-dependent q–f-coherent states at different instants.

1. Introduction

The coherent states introduced by Schrödinger [1] and Glauber [2] are eigenstates of the boson annihilation operator a, and have widespread applications in the fields of physics [3–7]. The even and odd coherent states [8], which are two orthonormalized eigenstates of the square a^2 of the operator a, play an important role in quantum optics [9–11]. The k orthonormalized eigenstates of the kth power a^k $(k \geqslant 1)$ were constructed and applied to quantum optics [12,13]. The notion of coherent states was extended to q-coherent states [14], which are eigenstates of the q-boson annihilation operator a_q. The q-coherent states were well studied and applied widely to quantum optics and mathematical physics [14–22]. The even and odd q-coherent states [23], defined as two orthonormalized eigenstates of the square a_q^2 of the operator a_q, have non-classical effects [24]. Moreover, the k orthonormalized eigenstates of the kth power a_q^k were well investigated and applied to quantum optics [25, 26].

Recently, there has been much interest in the study of nonlinear coherent states called f-coherent states [27], which are eigenstates of the annihilation operator $af(N)$ of f-oscillators, where $f(N)$ is an operator-valued function of the boson number operator N. A class of f-coherent states can be realized physically as the stationary states of the centre-of-mass motion of a trapped ion [28]. The f-coherent states exhibit non-classical features such as squeezing and self-splitting. Subsequently, the even and odd f-coherent states, which are two orthonormalized eigenstates of the square $(af(N))^2$ of the operator $af(N)$, were constructed and their non-classical effects were studied [29, 30]. In a previous paper [31], we obtained k orthonormalized eigenstates of the kth power $(af(N))^k$ and discussed their properties. Naturally, in this paper, it is very desirable to study the orthonormalized eigenstates of the kth power $(a_q f(N_q))^k$ of the operator $a_q f(N_q)$, where $f(N_q)$ is an operator-valued function of

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the q-boson number operator N_q. We refer to $a_q f(N_q)$ as the generalized q-boson annihilation operator.

The paper is organized as follows. In section 2, the k orthonormalized eigenstates of the operator $(a_q f(N_q))^k$ are obtained, and their properties are discussed. In section 3, an alternative method to construct them is proposed. Their physical meaning is explored in section 4.

2. The k orthonormalized eigenstates of $(a_q f(N_q))^k$

The q-boson annihilation operator a_q, creation operator a_q^+ and number operator N_q satisfy the quantum Heisenberg–Weyl algebra

$$a_q a_q^+ - q a_q^+ a_q = q^{-N_q} \tag{1}$$

$$[N_q,\ a_q] = -a_q \qquad [N_q,\ a_q^+] = a_q^+ \tag{2}$$

with q real and positive. The operators a_q, a_q^+ and N_q act in a Hilbert space with the q-occupation number basis $|n\rangle_q$ $(n = 0, 1, 2, \ldots)$, such that

$$a_q|0\rangle_q = 0 \qquad |n\rangle_q = \frac{(a_q^+)^n}{\sqrt{[n]!}}|0\rangle_q \tag{3}$$

where

$$[n]! = [n][n-1]\ldots[1] \qquad [0]! = 1 \tag{4}$$

$$[n] = \frac{q^n - q^{-n}}{q - q^{-1}}. \tag{5}$$

Their actions on the basis states are given by

$$a_q|n\rangle_q = \sqrt{[n]}|n-1\rangle_q \qquad a_q^+|n\rangle_q = \sqrt{[n+1]}|n+1\rangle_q \qquad N_q|n\rangle_q = n|n\rangle_q. \tag{6}$$

Let us consider the following states:

$$|\psi_j(\alpha, f)\rangle_k = C_j \sum_{n=0}^{\infty} \frac{\alpha^{kn+j}}{\sqrt{[kn+j]!}\, f(kn+j)!}|kn+j\rangle_q \tag{7}$$

with

$$f(kn+j)! = f(kn+j) f(kn+j-1)\ldots f(1) \qquad f(0)! = 1 \tag{8}$$

where k is a positive integer $(k = 1, 2, 3, \ldots)$; $j = 0, 1, \ldots, k-1$; C_j are normalized factors and α is a complex parameter. Let $A = a_q f(N_q)$. With the kth power A^k operating on $|\psi_j(\alpha, f)\rangle_k$, we have

$$A^k|\psi_j(\alpha, f)\rangle_k = \alpha^k|\psi_j(\alpha, f)\rangle_k. \tag{9}$$

As a result, the k states of (7) are all the eigenstates of the operator $(a_q f(N_q))^k$ with the same eigenvalue α^k. It is easy to check that, for the same value of k, these states are orthogonal to each other with respect to the subscript j, i.e.

$${}_k\langle\psi_i(\alpha, f)|\psi_j(\alpha', f)\rangle_k = 0 \qquad i, j = 0, 1, \ldots, k-1, i \neq j. \tag{10}$$

Let $|\alpha|^2 = x$. We easily suppose C_j to be a real number. Using the normalized conditions

$${}_k\langle\psi_j(\alpha, f)|\psi_j(\alpha, f)\rangle_k = C_j^2 \sum_{n=0}^{\infty} \frac{x^{kn+j}}{[kn+j]!|f(kn+j)!|^2} = C_j^2 A_j(x, f) = 1 \tag{11}$$

we have

$$C_j = A_j^{-\frac{1}{2}}(x, f) \tag{12}$$

where

$$A_j(x, f) = \sum_{n=0}^{\infty} \frac{x^{kn+j}}{[kn+j]!|f(kn+j)!|^2}. \tag{13}$$

From (13) it follows that

$$\sum_{j=0}^{k-1} A_j(x, f) = \sum_{n=0}^{\infty} \frac{x^n}{[n]!|f(n)!|^2} \equiv e_{q,f}(x). \tag{14}$$

It should be noted that the k states $|\psi_j(\alpha, f)\rangle_k$ $(j = 0, 1, \ldots, k-1)$ are normalizable provided C_j are non-zero and finite. This means that the terms in summation for $A_j(x, f)$ should be such that

$$|\alpha|^2 < \lim_{n\to\infty} [n]|f(n)|^2. \tag{15}$$

If $|f(n)|$ decreases faster than $[n]^{-\frac{1}{2}}$ for large n, then the range of α for which the states $|\psi_j(\alpha, f)\rangle_k$ are normalizable is restricted to values satisfying (15) and in other cases the range of α is unrestricted.

We may obtain

$$A|\psi_j(\alpha, f)\rangle_k = \alpha A_j^{-\frac{1}{2}}(|\alpha|^2, f) A_{j-1}^{\frac{1}{2}}(|\alpha|^2, f)|\psi_{j-1}(\alpha, f)\rangle_k \qquad j = 1, 2, \ldots, k-1 \tag{16}$$

$$A^i|\psi_0(\alpha, f)\rangle_k = \alpha^i A_0^{-\frac{1}{2}}(|\alpha|^2, f) A_{k-i}^{\frac{1}{2}}(|\alpha|^2, f)|\psi_{k-i}(\alpha, f)\rangle_k \qquad i = 1, 2, \ldots, k. \tag{17}$$

This indicates that, by the successive actions of the operator A, the k eigenstate vectors of A^k can be transformed into each other in this way: $|\psi_0\rangle_k \to |\psi_{k-1}\rangle_k \to |\psi_{k-2}\rangle_k \to \cdots \to |\psi_1\rangle_k \to |\psi_0\rangle_k$. Actually, the operator A plays the role of a rotating operator in the k eigenstate vectors of A^k.

According to (7), for $k = 1$, we obtain

$$|\psi_0(\alpha, f)\rangle_1 = e_{q,f}^{-\frac{1}{2}}(|\alpha|^2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!} f(n)!} |n\rangle_q \equiv |\alpha, f\rangle. \tag{18}$$

The states $|\alpha, f\rangle$ are eigenstates of $a_q f(N_q)$. This is a generalization of the notion of f-coherent states, which are eigenstates of $af(N)$. Therefore, we call $|\alpha, f\rangle$ the q–f-coherent states.

According to (7), for $k = 2$, we obtain

$$|\psi_0(\alpha, f)\rangle_2 = A_0^{-\frac{1}{2}}(|\alpha|^2, f) \sum_{n=0}^{\infty} \frac{\alpha^{2n}}{\sqrt{[2n]!} f(2n)!} |2n\rangle_q \tag{19}$$

$$|\psi_1(\alpha, f)\rangle_2 = A_1^{-\frac{1}{2}}(|\alpha|^2, f) \sum_{n=0}^{\infty} \frac{\alpha^{2n+1}}{\sqrt{[2n+1]!} f(2n+1)!} |2n+1\rangle_q. \tag{20}$$

The states $|\psi_0(\alpha, f)\rangle_2$ and $|\psi_1(\alpha, f)\rangle_2$ are two orthonormalized eigenstates of $(a_q f(N_q))^2$. This is a generalization of the notion of even and odd f-coherent states, which are two orthonormalized eigenstates of $(af(N))^2$. Therefore, we call $|\psi_0(\alpha, f)\rangle_2$ and $|\psi_1(\alpha, f)\rangle_2$ the even and odd q–f-coherent states, respectively.

In terms of the k eigenstates $|\psi_j(\alpha, f)\rangle_k$ of A^k, the q–f-coherent states can be expanded in this way:

$$|\alpha, f\rangle = e_{q,f}^{-\frac{1}{2}}(|\alpha|^2) \left[\sum_{j=0}^{k-1} A_j^{\frac{1}{2}}(|\alpha|^2, f)|\psi_j(\alpha, f)\rangle_k \right]. \tag{21}$$

Note that $|\alpha, f\rangle$ and $|\psi_j(\alpha, f)\rangle_k$ are non-trivially different.

We should emphasize that here we discuss orthogonality of $|\psi_j(\alpha, f)\rangle_k$ with respect to subscript j. For $\alpha \neq \alpha'$, we obtain

$$_k\langle\psi_j(\alpha, f)|\psi_j(\alpha', f)\rangle_k = [A_j(|\alpha|^2, f)A_j(|\alpha'|^2, f)]^{-\frac{1}{2}} A_j(\alpha^*\alpha', f) \neq 0. \tag{22}$$

Therefore, when $\alpha \neq \alpha'$, $|\psi_j(\alpha, f)\rangle_k$ and $|\psi_j(\alpha', f)\rangle_k$ are not orthogonal.

As three limiting cases, for $q\to 1$ $|\psi_j(\alpha, f)\rangle_k$ become k orthonormalized eigenstates of $(af(N))^k$; for $f(N_q)\to 1$ $|\psi_j(\alpha, f)\rangle_k$ become those of a_q^k; for $q\to 1$ and $f(N_q)\to 1$ $|\psi_j(\alpha, f)\rangle_k$ become those of a^k.

Now, we give some applications of the result (7). Taking $f(N_q)$ to be $\frac{1}{\sqrt{[N_q]}}$, we find k orthonormalized eigenstates of the kth power $(\exp(\mathrm{i}\varphi))^k$ of the q-photon phase operator $\exp(\mathrm{i}\varphi)(\equiv a_q \frac{1}{\sqrt{[N_q]}})$ [16], namely,

$$|\psi_j(\alpha, f)\rangle_k = \frac{\sqrt{1-|\alpha|^{2k}}}{|\alpha|^j}\sum_{n=0}^{\infty}\alpha^{kn+j}|kn+j\rangle_q \qquad |\alpha| < 1. \tag{23}$$

Taking $f(N_q)$ to be $\sqrt{[N_q]}$, we find k orthonormalized eigenstates of the kth power K_-^k of the annihilation operator $K_-(\equiv a_q\sqrt{[N_q]})$ of the quantum $SU(1,1)_q$ algebra in the Holstein–Primakoff realization [15], namely,

$$|\psi_j(\alpha, f)\rangle_k = A_j^{-\frac{1}{2}}\sum_{n=0}^{\infty}\frac{\alpha^{kn+j}}{[kn+j]!}|kn+j\rangle_q \qquad |\alpha| < \infty \tag{24}$$

with

$$A_j = \sum_{n=0}^{\infty}\frac{(|\alpha|^2)^{kn+j}}{([kn+j]!)^2}. \tag{25}$$

It is noteworthy that Klauder and co-workers have studied an extremely wide class of coherent states that includes the f-coherent states as a small subset [32, 33]. However, the k orthonormalized eigenstates of $(a_q f(N_q))^k$ are different from the Klauder-type coherent states. In the limiting case $q\to 1$, the k states can also be obtained by considering a suitable linear superposition of the Klauder-type states.

3. An alternative method for construction of the k orthonormalized eigenstates of $(a_q f(N_q))^k$

According to (18), we consider the following k q–f-coherent states:

$$\begin{aligned}|\alpha_l, f\rangle &= |\alpha \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}l}, f\rangle \\ &= e_{q,f}^{-\frac{1}{2}}(|\alpha|^2)\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{[n]!}f(n)!}\mathrm{e}^{\mathrm{i}\frac{2\pi}{k}ln}|n\rangle_q \qquad l = 0, 1, \ldots, k-1.\end{aligned} \tag{26}$$

The k q–f-coherent states are discretely distributed with an equal interval of angle along a circle around the origin of the α-plane. The inner product of the two states of (26) is

$$\langle\alpha_l, f|\alpha_{l'}, f\rangle = e_{q,f}^{-1}(|\alpha|^2)e_{q,f}(|\alpha|^2\mathrm{e}^{\mathrm{i}\frac{2\pi}{k}(l'-l)}) \qquad l, l' = 0, 1, \ldots, k-1. \tag{27}$$

Consider a linear transformation S such that

$$|\varphi\rangle_k = S|\alpha, f\rangle_k \tag{28}$$

where

$$|\alpha, f\rangle_k = \begin{bmatrix} |\alpha_0, f\rangle \\ |\alpha_1, f\rangle \\ \vdots \\ |\alpha_{k-1}, f\rangle \end{bmatrix} \qquad |\varphi\rangle_k = \begin{bmatrix} |\varphi_0\rangle_k \\ |\varphi_1\rangle_k \\ \vdots \\ |\varphi_{k-1}\rangle_k \end{bmatrix}. \tag{29}$$

S is a $k \times k$ matrix that makes φ_j orthonormal, and ${}_k\langle\varphi_j|\varphi_{j'}\rangle_k = \delta_{jj'}$. The above requirement leads to a set of algebraic equations for S_{ij},

$$\sum_{l=0}^{k-1}\sum_{l'=0}^{k-1} e_{q,f}^{-1}(|\alpha|^2) e_{q,f}(|\alpha|^2 \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}(l'-l)}) S^*_{jl} S_{j'l'} = \delta_{jj'}. \tag{30}$$

The solution of equation (30), S_{ij}, can be found as follows. By virtue of the relation

$$\sum_{l'=0}^{k-1} e_{q,f}(|\alpha|^2 \mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{k}(l'-l)}) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl'} = \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl} \sum_{l'=0}^{k-1} e_{q,f}(|\alpha|^2 \mathrm{e}^{\pm\mathrm{i}\frac{2\pi}{k}l'}) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl'} \tag{31}$$

the matrix elements of S that satisfy (30) are given by

$$\begin{aligned} S_{jl} &= \frac{1}{k} e_{q,f}^{\frac{1}{2}}(|\alpha|^2) \left[\frac{1}{k}\sum_{l'=0}^{k-1} e_{q,f}(|\alpha|^2 \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}l'}) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl'}\right]^{-\frac{1}{2}} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl} \\ &= \frac{1}{k} e_{q,f}^{\frac{1}{2}}(|\alpha|^2) A_j^{-\frac{1}{2}}(|\alpha|^2, f) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl} \qquad j, l = 0, 1, \ldots, k-1. \end{aligned} \tag{32}$$

From (28) and (32), we obtain k orthonormalized states

$$|\varphi_j\rangle_k = \frac{1}{k} A_j^{-\frac{1}{2}}(|\alpha|^2, f) e_{q,f}^{\frac{1}{2}}(|\alpha|^2) \sum_{l=0}^{k-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl} |\alpha \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}l}, f\rangle \qquad j = 0, 1, \ldots, k-1 \tag{33}$$

which are just what we want. By use of the relation

$$\sum_{l=0}^{k-1} \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}lt} = 0 \qquad t = 1, 2, \ldots, k-1 \tag{34}$$

it can be proved that

$$|\varphi_j\rangle_k = |\psi_j(\alpha, f)\rangle_k \qquad j = 0, 1, \ldots, k-1. \tag{35}$$

According to (33), for $k = 2$, we obtain

$$|\varphi_0\rangle_2 = \tfrac{1}{2} A_0^{-\frac{1}{2}}(|\alpha|^2, f) e_{q,f}^{\frac{1}{2}}(|\alpha|^2)(|\alpha, f\rangle + |-\alpha, f\rangle) \tag{36}$$

$$|\varphi_1\rangle_2 = \tfrac{1}{2} A_1^{-\frac{1}{2}}(|\alpha|^2, f) e_{q,f}^{\frac{1}{2}}(|\alpha|^2)(|\alpha, f\rangle - |-\alpha, f\rangle). \tag{37}$$

This indicates that the even and odd q–f-coherent states can be represented as a linear superposition of two q–f-coherent states, which have the same amplitude but opposite phases.

The states $|\varphi_j\rangle_k$ ($j = 0, 1, \ldots, k-1$) in (33) are exactly the k orthonormalized eigenstates of $(a_q f(N_q))^k$ obtained in section 2, but reconstructed here by a different method. From the above reconstruction, we come to an important conclusion that any orthonormalized eigenstates of $(a_q f(N_q))^k$ can be expressed as a linear superposition of k q–f-coherent states $|\alpha \mathrm{e}^{\mathrm{i}\frac{2\pi}{k}l}, f\rangle$ ($l = 0, 1, \ldots, k-1$), which have the same amplitude but different phases. Yet, from (33), one can find the connection between q–f-coherent states and these k eigenstates.

It is interesting to note that the above discussion includes two limiting cases of $f(N_q) \to 1$ and $q \to 1$ studied in [25] and [31], respectively.

4. Physical meaning of the k orthonormalized eigenstates of $(a_q f(N_q))^k$

In this section, we shall explore the physical meaning of the k orthonormalized eigenstates of $(a_q f(N_q))^k$ by constructing them from the time-dependent q–f-coherent states generated from a time-dependent Schrödinger equation.

Suppose a system evolves according to the time-dependent Schrödinger equation

$$\mathrm{i}\hbar\frac{\partial}{\partial t}|\chi(t)\rangle = H|\chi(t)\rangle. \tag{38}$$

If the system is initially ($t = 0$) in a q–f-coherent state $|\alpha, f\rangle$, and the Hamiltonian is independent of time, then at time t the system reaches the state

$$|\chi(t)\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar}Ht}|\alpha, f\rangle. \tag{39}$$

Choosing $H = \hbar\omega N_q$, we have

$$|\chi(t)\rangle = |\alpha\mathrm{e}^{-\mathrm{i}\omega t}, f\rangle. \tag{40}$$

Therefore, at the instant

$$t_l = \frac{2\pi}{\omega}\frac{l}{k} \qquad k = 1, 2, 3, \ldots \qquad l = 0, 1, \ldots, k-1 \tag{41}$$

the system is in the state

$$|\chi(t_l)\rangle = |\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}, f\rangle. \tag{42}$$

Now let us consider a linear superposition of the above time-dependent q–f-coherent states at different instants,

$$|\phi_i\rangle = \sum_{l=0}^{k-1} C_l^i|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}, f\rangle. \tag{43}$$

Suitably choosing the expansion coefficients, we can construct the k orthonormalized states. The inner product of the two states of (43) is

$$\begin{aligned}\langle\phi_i|\phi_j\rangle &= \sum_{l=0}^{k-1}\sum_{l'=0}^{k-1}(C_l^i)^* C_{l'}^j\langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}, f|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l'}, f\rangle \\ &= \langle C_i|\tilde{M}|C_j\rangle \qquad i, j = 0, 1, \ldots, k-1\end{aligned} \tag{44}$$

where

$$|C_j\rangle = \begin{bmatrix} C_0^j \\ C_1^j \\ \vdots \\ C_{k-1}^j \end{bmatrix} \tag{45}$$

$$\langle C_i| = [C_0^i C_1^i \ldots C_{k-1}^i]^* \tag{46}$$

$$\tilde{M} = \begin{bmatrix} \langle\alpha|\alpha\rangle & \langle\alpha|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}\rangle & \cdots & \langle\alpha|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}\rangle \\ \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}|\alpha\rangle & \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}\rangle & \cdots & \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}\rangle \\ \vdots & \vdots & \vdots & \vdots \\ \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}|\alpha\rangle & \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}}\rangle & \cdots & \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(k-1)}\rangle \end{bmatrix}. \tag{47}$$

Note that the symbol f is suppressed in the expression of the matrix elements of $\tilde{M}$, i.e.

$$\langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l'}\rangle \equiv \langle\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}, f|\alpha\mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l'}, f\rangle \qquad l, l' = 0, 1, \ldots, k-1.$$

Thus, the matrix elements of $\tilde{M}$ read

$$\tilde{M}_{l,l'} = e^{-1}_{q,f}(|\alpha|^2) e_{q,f}(|\alpha|^2 \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}(l'-l)}). \tag{48}$$

Because $\tilde{M}$ is Hermitian, its eigenstates with different eigenvalues must be orthogonal to one another. Suppose that $|C_i\rangle$ and $|C_j\rangle$ are its two eigenstates. It follows that

$$\begin{aligned} \tilde{M}|C_i\rangle &= \lambda_i |C_i\rangle \\ \tilde{M}|C_j\rangle &= \lambda_j |C_j\rangle \end{aligned} \tag{49}$$

where

$$|C_j\rangle = \begin{bmatrix} 1 \\ \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}j} \\ \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}j2} \\ \vdots \\ \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}j(k-1)} \end{bmatrix} \tag{50}$$

and

$$\lambda_j = e^{-1}_{q,f}(|\alpha|^2) \sum_{l=0}^{k-1} e_{q,f}(|\alpha|^2 \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}) \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl}. \tag{51}$$

The orthonormality relation reads

$$\langle C_i|C_j\rangle = k\delta_{ij}. \tag{52}$$

Replacing the expansion coefficients in (43) by the column vector (50) and considering the normalization condition of the states (43), we obtain

$$|\phi_j\rangle = (k\lambda_j)^{-\frac{1}{2}} \sum_{l=0}^{k-1} \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl} |\alpha \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}l}, f\rangle \qquad j = 0, 1, \ldots, k-1. \tag{53}$$

By virtue of (49) and (52), it is easy to prove that the inner product of two states of (53) is

$$\langle\phi_i|\phi_j\rangle = \frac{(\lambda_i\lambda_j)^{-\frac{1}{2}}}{k} \langle C_i|\tilde{M}|C_j\rangle = \left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1}{2}} \delta_{ij} = \delta_{ij} \tag{54}$$

which indicates that the states (53) form an orthonormalized set.

The physical meaning of the state $|\phi_j\rangle$ in (53) has been made clearer. The state $|\phi_j\rangle$ can be generated by a linear superposition of the k time-dependent q–f-coherent states $|\alpha \mathrm{e}^{-\mathrm{i}\omega t_l}, f\rangle$ $(l = 0, 1, \ldots, k-1)$ at different instants, while the superposition coefficients are C_l^j $(= \mathrm{e}^{-\mathrm{i}\frac{2\pi}{k}jl})$.

It can be proved that

$$|\phi_0\rangle = |\psi_0(\alpha, f)\rangle_k \tag{55}$$

$$|\phi_{k-l}\rangle = |\psi_l(\alpha, f)\rangle_k \qquad l = 1, 2, \ldots, k-1. \tag{56}$$

Therefore, $|\phi_j\rangle_k$ $(j = 0, 1, \ldots, k-1)$ are exactly the k orthonormalized eigenstates of $(a_q f(N_q))^k$ in (7).

The above discussion includes the limiting case of $f(N_q) \to 1$ investigated in [25]. In fact, the method for construction of the k eigenstates of $(a_q f(N_q))^k$ in this section is somewhat different from that in section 3.

5. Conclusions

We have derived the k orthonormalized eigenstates of the kth power $(a_q f(N_q))^k$ $(k \geqslant 1)$ of the generalized q-boson annihilation operator $a_q f(N_q)$, discussed their properties and given some applications of the result. We have proposed an alternative method to construct these eigenstates of $(a_q f(N_q))^k$, and come to an important conclusion that all of them can be expressed as a linear superposition of k $q{-}f$-coherent states that have the same amplitude but different phases. We have also explored their physical meaning and shown that they can be generated by a linear superposition of the time-dependent $q{-}f$-coherent states at different instants.

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